MATHEMATICS

SOME ANALOGIES BETWEEN COMMUTATIVE RINGS, RIESZ SPACES AND DISTRIBUTIVE LATTICES WITH SMALLEST ELEMENT

BY

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(Communicated by Prof. A. C. Zaanen at the meeting of October 27, 1973)

1. Boolean rings, regular rings and hyper-archimedean Riesz spaces

By a well-known theorem of M. H. Stone ([11], theorem 18), a distributive lattice X with smallest element θ is a Boolean ring if and only if every proper prime ideal in X is a maximal ideal. It is not difficult to prove that each of these two conditions for X is equivalent to requiring that every principal ideal in X is, so to speak, a direct summand, i.e., if $x \in X$ and $A_x = (y: y \in X, \theta \leqslant y \leqslant x)$ denotes the principal ideal generated by x, then there exists an ideal J_x, depending on x, such that

$$X = A_x \vee J_x \text{ and } A_x \cap J_x = \{\theta\}.$$

Necessarily, J_x is then equal to A_x^d, the disjoint complement of A_x. We indicate the simple proof. Assume first that $X = A_x \vee A_x^d$ for all $x \in X$. We have to prove that every proper prime ideal is a maximal ideal. If P is a proper prime ideal and $x \notin P$, then $A_x^d \subset P$, and therefore

$$X = A_x \vee A_x^d \subset (x, P),$$

where $(x, P) = A_x \vee P$ denotes the ideal generated by x and P. Hence, $(x, P) = X$ and thus every ideal Q that properly contains P satisfies $Q = X$. In other words, P is a maximal ideal.

Conversely, suppose that every proper prime ideal in X is a maximal ideal and let $x \in X$. If $X = A_x \vee A_x^d$ does not hold, then $A_x \vee A_x^d$ is contained in some proper prime ideal P. From the hypothesis it follows that every proper prime ideal in X is a minimal prime ideal. But a minimal prime ideal cannot contain an element and its disjoint complement simultaneously, whereas P contains x and its disjoint complement. We have obtained, therefore, a contradiction. Hence, $X = A_x \vee A_x^d$ for all $x \in X$.

One of the main purposes of this paper is to investigate whether these

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[1]) The contents of this paper are partly derived from the author's doctoral thesis ([12]) at Leiden university.

three conditions are also equivalent in a commutative ring and in a Riesz space.

First, let us consider a commutative ring R. If R has a unit element, then every principal ideal in R is a direct summand (this means that for every $a \in R$ there exists an ideal J_a, depending on a, such that $R=(a) \oplus J_a$, where (a) denotes the principal ideal generated by a) if and only if R is regular (in the sense of J. von Neumann). We recall that a commutative ring R is called (von Neumann) regular if for every $a \in R$ there exists an element $r \in R$ such that $a=a^2r$. However, if R does not have a unit element, the condition that every principal ideal in R is a direct summand does not necessarily imply that R is regular (take e.g. for R the additive group of residue classes modulo p (p prime number) with the trivial multiplication). However, if we impose upon R the additional condition that R is semi-prime, that is to say that the only nilpotent element in R is the zero element, then the condition that every principal ideal in R is a direct summand is equivalent to regularity of R. Moreover, we have necessarily in this case that $R=(a) \oplus I(a)$ for all $a \in R$, where $I(a)$ denotes the annihilator of a. More precisely, the following theorem holds.

THEOREM 1. *In a commutative ring R the following conditions are equivalent.*

(1) *R is regular.*
(2) *R is semi-prime and every principal ideal in R is a direct summand.*
(3) *R is semi-prime and every proper prime ideal in R is a maximal ideal.*

Essentially, the equivalence of (1) and (3) is due to R. HAMSHER (see [6], section 1 and [7], section 2.2, exercises 12 and 22). Since in the books referred to only brief indications of the equivalence of (1) and (3) are given, we present here a complete proof of the theorem.

PROOF. (1) $\Rightarrow$ (2). Since the regularity of R is equivalent to saying that $A=\sqrt{A}$ holds for all ideals A in R, where $\sqrt{A}$ denotes the radical of A (for the proof we refer to the book by N. H. MCCOY, [10], theorem 49), we obtain in particular that $\sqrt{(0)}=(0)$, i.e., R is semi-prime. Also, if a is an arbitrary element of R and $r \in R$ satisfies $a=a^2r$, it follows from

$$s=sar+(s-sar) \text{ and } a(s-sar)=0$$

for all $s \in R$ that $R=(a)+I(a)$. Combining this with $(a) \cap I(a)=(0)$, which is an easy consequence of the fact that R is semi-prime, we find $R=(a) \oplus I(a)$. It follows that (a) is a direct summand for all $a \in R$.

(2) $\Rightarrow$ (3). Let P be a proper prime ideal in R and let $a \notin P$. Then $I(a) \subset P$. By hypothesis, there exists an ideal J_a such that $R=(a) \oplus J_a$. Since $ja \in (a) \cap J_a=(0)$ for all $j \in J_a$, we have $J_a \subset I(a)$ (in fact, it is

easy to prove that in the present circumstances $J_a = I(a)$). Hence

$$R = (a) \oplus J_a \subset (a, P),$$

where (a, P) denotes the ideal generated by a and P. This shows that $(a, P) = R$ and thus any ideal Q that properly contains P satisfies $Q = R$. In other words, P is a maximal ideal.

(3) $\Rightarrow$ (1). The proof of this implication is divided into several parts.

(i) Given $a \in R$, we have to prove that there exists $r \in R$ such that $a = a^2 r$. We may assume $a \neq 0$, so $I(a) \neq R$ (observe that R is semi-prime and therefore $a \notin I(a)$). Hence, the quotient ring $S = R/I(a)$ does not consist exclusively of the zero element. Although R is not supposed to have a unit element, we will show in the following parts that the commutative ring S in the present circumstances has a unit element. Let us denote the canonical mapping of R onto S by ϕ and the image $\phi(t)$ of an element $t \in R$ by $\bar{t}$.

(ii) Note first that S is semi-prime. For the proof, let $\bar{s}^n = \bar{0}$ for some $\bar{s} \in S$ and some natural number n. This implies $s^n \in I(a)$, so $as^n = 0$. It follows that $a^n s^n = 0$, and therefore $as = 0$ since R is semi-prime. Hence, $s \in I(a)$, i.e., $\bar{s} = \bar{0}$. Moreover, the ring S has at least one element that is not a zero-divisor (for example, the element $\bar{a}$). Indeed, $\bar{a} \neq \bar{0}$, since it would follow otherwise from $a \in I(a)$ that $a^2 = 0$, so $a = 0$. Additionally, $\bar{a}\bar{b} = \bar{0}$ implies $ab \in I(a)$ and thus $a^2 b = 0$. But then $a^2 b^2 = 0$ implies $ab = 0$, i.e., $\bar{b} = \bar{0}$. Let now T be the subset of S, defined by

$$T = (c^n - c^n s;\ s \in S,\ n = 1, 2, \ldots),$$

where c is a fixed non-zero-divisor in S. If the zero element $\bar{0}$ of S is in T, then S has a unit element. Indeed, it follows from $c^n = c^n s_0$ for some $s_0 \in S$ and some natural number n that $c^n\ (s - ss_0) = \bar{0}$ for all $s \in S$, so we obtain $s = ss_0$ for all $s \in S$, using that $c^n \neq \bar{0}$ and the fact that c^n is not a zero-divisor either. In this case the element s_0 is, therefore, the unit element of S.

(iii) Hence, for the proof that S has a unit element it remains to show that $\bar{0} \in T$. Assume on the contrary that $\bar{0} \notin T$, i.e., the intersection $(\bar{0}) \cap T$ is empty. Since T is multiplicatively closed, there exists a prime ideal P in S such that $P \cap T$ is empty (take for P an ideal maximal with respect to the property that its intersection with T is empty). But then $\phi^{-1}(P)$ is a proper prime ideal in R and thus, by hypothesis, $\phi^{-1}(P)$ is a maximal ideal. Therefore, $R/\phi^{-1}(P)$ is a field (this is not trivial since R does not necessarily have a unit element; we use here theorem 19 of [10]). By a well-known ring theoretical theorem, $R/\phi^{-1}(P)$ and S/P are ring isomorphic, so S/P is also a field. Let $\tilde{e}$ be the unit element of this field and let e be an element of S in the residue class $\tilde{e}$. Then we have $\tilde{c} = \tilde{c}\tilde{e}$, so $c - ce \in P$, which contradicts the fact that $P \cap T$ is empty. It follows that $S = R/I(a)$ has a unit element. Call this element $\bar{e}$.

(iv) It is easy to check that the hypothesis implies that every proper prime ideal in S is a minimal prime ideal. Since $\bar{a}$ is a non-zero-divisor in S, the element $\bar{a}$ is not a member of any minimal prime ideal in S (cf. [4], corollary 2.5). Hence, $\bar{a}$ is not contained in any proper prime ideal of S. But this implies that $\bar{a}$ is a unit in S (i.e., the ideal generated by $\bar{a}$ in S is the whole of S), since otherwise $\bar{a}$ would belong to some maximal (and hence proper prime) ideal. Note that the existence of a unit element in S is used here. Therefore, there exists an element $\bar{r} \in S$ such that $\bar{e} = \bar{a}\bar{r}$, which implies $\bar{a} = \bar{a}^2\bar{r}$, i.e., $a - a^2r \in I(a)$. On the other hand $a - a^2r \in (a)$, so $(a) \cap I(a) = (0)$ implies $a = a^2r$, which finishes the proof.

Exactly the same theorem holds for Riesz spaces. Before stating this theorem we first recall that a Riesz space L is called hyper-archimedean whenever every principal ideal in L is a direct summand, i.e., if u is an element of the positive cone L^+ of L, and A_u denotes the principal ideal generated by u, then there exists an ideal J_u, depending on u, such that $L = A_u \oplus J_u$. Necessarily, J_u is again the disjoint complement $A_u{}^d$ of A_u. The space L is called hyper-archimedean in this case since this property is equivalent to the property that L/A is Archimedean for all ideals A in L (see [9], theorem 37.6).

The theorem for Riesz spaces that corresponds to theorem 1 is known. I. Amemiya makes mention of it, although briefly and without complete proofs, in one of his papers (see [1], the final paragraphs of section 6), and a proof using some results concerning the hull-kernel topology in the collection of all proper prime ideals in L can be found in the book by W. A. J. Luxemburg and A. C. Zaanen ([9], theorem 37.6). The proof we shall present here is direct and is, in fact, a copy of the proof of theorem 1 in the following sense.

When one compares the multiplication in a commutative ring R with the infimum operation in a Riesz space L (or in a distributive lattice X with smallest element), the notions of ideal and prime ideal in these structures are very much alike. Also, annihilator in R and disjoint complement in L (or in X) are corresponding notions. Keeping in mind this point of view, it is not difficult to copy the proof of theorem 1 for Riesz spaces. We merely have to replace multiplication by the infimum operation and annihilator by disjoint complement. We shall now state and prove this theorem. Note that its proof is independent of any fact concerning the hull-kernel topology.

Theorem 2. *In a Riesz space L the following conditions are equivalent.*
(1) *L is hyper-archimedean.*
(2) *Every proper prime ideal in L is a maximal ideal.*

Proof. (1) $\Rightarrow$ (2). Let P be a proper prime ideal and let $0 \leqslant u \notin P$. Then $A_u{}^d \subset P$, so

$$L = A_u \oplus A_u{}^d \subset (u, P),$$

where $(u, P) = A_u + P$ denotes the ideal generated by u and P. Therefore, $(u, P) = L$. It follows that any ideal Q that properly contains P satisfies $Q = L$, and hence P is a maximal ideal.

(2) $\Rightarrow$ (1). Let u be an arbitrary element of L^+, and let us denote the equivalence class of u modulo $A_u{}^d$ by $\bar{u}$. Observe first that $\{\bar{u}\}^d = \{\bar{0}\}$ in $L/A_u{}^d$. Indeed, $\bar{0} \leqslant \bar{v} \in \{\bar{u}\}^d$ implies $\inf(\bar{u}, \bar{v}) = \bar{0}$, in other words $\inf(u, v) \in A_u{}^d$. On the other hand $\inf(u, v)$ is a member of A_u, so $\inf(u, v) = 0$, i.e., $\bar{v} = \bar{0}$. From $\{\bar{u}\}^d = \{\bar{0}\}$ it follows that $\bar{u}$ is not a member of any minimal prime ideal in $L/A_u{}^d$ (cf. [9], exercise 33.9).

It is an easy consequence of the hypothesis that every proper prime ideal in $L/A_u{}^d$ is a minimal prime ideal, and therefore the element $\bar{u}$ does not belong to any proper prime ideal in $L/A_u{}^d$. But then $\bar{u}$ is a strong unit in $L/A_u{}^d$, i.e., the principal ideal generated by $\bar{u}$ in $L/A_u{}^d$ is the whole of $L/A_u{}^d$.

Hence, if w is an arbitrary element of L^+, then $\bar{w}$ is majorized by a multiple of $\bar{u}$. In other words, $\bar{w} = \inf(\bar{w}, k\bar{u})$ for some natural number k, i.e.,

$$w - \inf(w, ku) \in A_u{}^d.$$

Writing $w_1 = \inf(w, ku)$, we get

$$w = w_1 + w_2, \quad w_1 \in A_u, \quad w_2 \in A_u{}^d.$$

This means that every positive element of L is the sum of an element of A_u and an element of $A_u{}^d$. It follows immediately that $L = A_u \oplus A_u{}^d$.

Remark. Another way of proving the above implication (2) $\Rightarrow$ (1) is the following. If for some $u \in L^+$ the equality $L = A_u \oplus A_u{}^d$ does not hold, then the ideal $A_u \oplus A_u{}^d$ is contained in some proper prime ideal P. From the hypothesis it follows that P is a minimal prime ideal. But then P cannot contain u and the disjoint complement of u simultaneously (see, for example, [9], theorem 33.7 (iii)). It follows that $L = A_u \oplus A_u{}^d$ holds for all $u \in L^+$. Compare this proof with the corresponding proof for distributive lattices with smallest element.

From the foregoing it is evident that Boolean rings and hyper-archimedean Riesz spaces are closely related to each other. Indeed, both are characterized by the fact that every principal ideal is a direct summand. Now, a distributive lattice X with smallest element θ is a Boolean ring if and only if every principal ideal in X is a Boolean algebra, i.e., if and only if it follows from $\theta \leqslant y \leqslant x$ that there exists $z \in X$ such that

$$A_x = A_y \vee A_z \text{ and } A_y \cap A_z = \{\theta\},$$

where A_x, A_y and A_z denote the principal ideals generated by x, y and z respectively. It is a natural question to ask whether a corresponding characterization holds for hyper-archimedean Riesz spaces. This is indeed the case, as shown by the following theorem.

THEOREM 3. *Let L be a Riesz space. Then the following conditions are equivalent.*

(1) *L is hyper-archimedean.*

(2) *If $u \in L^+$ and $0 \leqslant v \leqslant u$, there exists $w \in L^+$ such that $A_u = A_v \oplus A_w$.*

PROOF. (1) $\Rightarrow$ (2). Let $u \in L^+$ and $0 \leqslant v \leqslant u$. Since $L = A_v \oplus A_v{}^d$, the element u can be written in the form $u = u' + w$ with $0 \leqslant u' \in A_v$, $0 \leqslant w \in A_u \cap A_v{}^d$. Since L is hyper-archimedean, every principal ideal in L is a projection band. In particular, $A_v = B_v$, where B_v denotes the band generated by v. By an elementary theorem on Riesz spaces with the principal projection property (see [9], corollary 31.2 (ii)) we have $B_v = B_{u'}$, showing that $A_v = A_{u'}$. It is an easy consequence of the Riesz decomposition property that $A_u = A_v \oplus (A_u \cap A_v{}^d)$. We assert that $A_w = A_u \cap A_v{}^d$ (if this can be shown to be true the proof of this part is complete). Evidently, $A_w \subset A_u \cap A_v{}^d$. To prove the converse, let $0 \leqslant z \in A_u \cap A_v{}^d$. Then $0 \leqslant z \leqslant ku = ku' + kw$ for an appropriate natural number k. Since $A_v{}^d = A_{u'}{}^d$, we have $\inf(z, u') = 0$. It follows from the Riesz decomposition property that

$$z = z_1 + z_2,\ \ 0 \leqslant z_1 \leqslant ku' \text{ and } 0 \leqslant z_2 \leqslant kw,$$

so

$$0 = \inf(z, u') \leqslant \inf(z_1, u') + \inf(z_2, u') = \inf(z_1, u') \leqslant \inf(z, u') = 0$$

(observe that $\inf(u', w) = 0$). Hence, $\inf(z_1, u') = 0$, which implies $z_1 = 0$. It follows that $z = z_2 \in A_w$. This concludes the proof.

(2) $\Rightarrow$ (1). Note first that if $w \in L^+$ and $A_u = A_v \oplus A_w$ $(0 \leqslant v \leqslant u)$, then $A_w = A_u \cap A_v{}^d$. In order to prove that L is hyper-archimedean it is sufficient to prove that every $u \in L^+$ has a projection on A_v for all $v \in L^+$, i.e., u can be written as

$$u = u_1 + u_2,\ \ 0 \leqslant u_1 \in A_v \text{ and } 0 \leqslant u_2 \in A_v{}^d.$$

We consider three cases.

(i) $0 \leqslant u \leqslant v$; take $u_1 = u$ and $u_2 = 0$.

(ii) $u \geqslant v \geqslant 0$; from the hypothesis it follows that $A_u = A_v \oplus A_w$ for some $w \in L^+$. Hence,

$$u = u_1 + u_2,\ \ 0 \leqslant u_1 \in A_v,\ \ 0 \leqslant u_2 \in A_w = A_u \cap A_v{}^d.$$

(iii) $u \in L^+$ and $v \in L^+$ arbitrary; from (i) it follows that $\inf(u, v)$ has a projection on A_v and from (ii) it follows that $\sup(u, v)$ has a projection on A_v. This implies that $u + v = \sup(u, v) + \inf(u, v)$ has also a projection on A_v. Therefore, $u + v - v = u$ has a projection on A_v. This concludes the proof.

2. Another characterization of regular rings

In spite of the beautiful correspondence between commutative regular rings, hyper-archimedean Riesz spaces and Boolean rings, there are properties that an arbitrary Riesz space L and an arbitrary distributive lattice X with smallest element have, but an arbitrary commutative ring R in general does not have. In the present section we show that, compared to Riesz spaces and distributive lattices with smallest element, the commutative regular rings in some respects play a very special role.

If A is an ideal in L or in X, then

$$A = \cap\,(P: P \text{ prime ideal, } P \supset A).$$

In R, however, we have

$$\sqrt{A} = \cap\,(P: P \text{ prime ideal, } P \supset A).$$

It follows that in R every ideal is equal to the intersection of all prime ideals containing the ideal if and only if $A = \sqrt{A}$ holds for all ideals A in R, i.e., if and only if R is regular.

Also, every ideal in L or in X, maximal with respect to the property of not containing a given element, is a prime ideal. Again, this property does not hold, in general, in an arbitrary commutative ring R (for example, let $R = K[Y]$, the polynomial ring in one variable Y over a field K; the ideal generated by Y^2 is not prime, but maximal with respect to the property of not containing Y). It is natural to ask in what kind of commutative rings this property holds. It turns out that commutative regular rings are precisely characterized by this property.

Theorem 4. *If R is a commutative ring, then R is regular if and only if every ideal in R, maximal with respect to the property of not containing a given element, is a prime ideal.*

Proof. First, suppose that R is a regular ring. Let P be an ideal in R, maximal with respect to the property of not containing a given element $r \in R$. If P is not prime, there exist elements $a, b \in R$ such that $ab \in P$, but $a \notin P$ and $b \notin P$. Since P is properly included in the ideal (P, a) generated by P and a, the element r is a member of (P, a). Hence,

$$r = p + sa + ka$$

for appropriate $p \in P$, $s \in R$ and integer k. Similarly,

$$r = q + tb + lb$$

for appropriate $q \in P$, $t \in R$ and integer l. An easy calculation shows that $r^2 \in P$. On account of the regularity of R there exists an element $x \in R$ satisfying $r = r^2x$. But this implies $r \in P$, a contradiction. Hence, P is a prime ideal.

Conversely, assume that every ideal in R, maximal with respect to the property of not containing a given element, is a prime ideal. We have to prove that, for an arbitrary $a \in R$, the element a belongs to the ideal

$$I_a = (a^2 s:\ s \in R).$$

Suppose on the contrary that $a \notin I_a$. By Zorn's lemma there exists an ideal $P \supset I_a$, maximal with respect to the property of not containing a. By hypothesis, P is prime. Observing that $a^3 \in I_a$, we find $a^3 \in P$, so $a \in P$, a contradiction. It follows that $a \in I_a$ and therefore $a = a^2 r$ for some $r \in R$. This is the desired result.

3. The radical of a Riesz space with a strong unit

The present section is, among other things, devoted to showing that the proof of a certain well-known theorem about the radical of a Riesz space with a strong unit can be simplified merely by using a corresponding result about the radical of a commutative ring with unit element.

It is well-known that in a commutative ring R with unit element e the Jacobson radical I_R of R, i.e.,

$$I_R = \cap\,(J: J \text{ maximal ideal in } R),$$

can be characterized as the set of all elements $x \in R$ such that $e - rx$ is a unit for all $r \in R$. Modifying this a little, we get

$$I_R = (x: x \in R,\ (e-y) = R \text{ for all } y \in (x)).$$

If I_L is the radical of a Riesz space L with a strong unit $e > 0$, i.e.,

$$I_L = \cap\,(J: J \text{ maximal ideal in } L),$$

then exactly the same argument as used in the proof of the characterization of the Jacobson radical shows that

$$(1) \qquad I_L = (f: f \in L,\ A_{e-h} = L \text{ for all } h \in A_f).$$

Indeed, suppose that $f \in I_L$ and that for some $h \in A_f$ the equality $A_{e-h} = L$ does not hold. Then A_{e-h} is, as a proper ideal, contained in some maximal ideal J. Since $f \in J$ and $h \in A_f$, we have $h \in J$. On the other hand $e - h \in J$. This implies $e \in J$, which is impossible. Conversely, assume that $A_{e-h} = L$ holds for all $h \in A_f$ and that f is not a member of some maximal ideal J.

It follows that the ideal (J, f) generated by J and f is the whole of L, i.e.,

$$(J, f) = J + A_f = L.$$

In particular, $e = j_0 + h_0$ for appropriate $j_0 \in J$, $h_0 \in A_f$. Hence, $e - h_0 \in J$. By hypothesis, $A_{e-h_0} = L$, so $J = L$, a contradiction.

By means of the characterization of I_L as given in formula (1), we are able to give a very simple proof of a theorem due to M. Fukamiya and K. Yosida (see [3] and also [9], theorem 27.5, for a different proof), stating that I_L is the ideal of all infinitely small elements in L. We recall that an element $f \in L$ is said to be infinitely small whenever there exists an element $g \in L$ such that $n|f| \leqslant |g|$ for all natural numbers n, or, equivalently, whenever $n|f| \leqslant e$ for all natural numbers n.

THEOREM 5. *If L is a Riesz space with a strong unit $e > 0$, then*

$$I_L = (f : f \in L,\ n|f| \leqslant e \text{ for } n = 1, 2, \ldots). \tag{2}$$

PROOF. Writing I for the ideal of all infinitely small elements in L, we have to prove that $I = I_L$. First, we shall prove that $I_L \subset I$. To this end, take $f \in I_L$ and suppose $f \notin I$. Then there exists a natural number k such that $k|f| \leqslant e$ fails to hold. If $k|f| \geqslant e$ holds, then $e \in A_f$, so, by (1), $A_{e-e} = \{0\} = L$. This is impossible since L contains the non-zero element e. It is also possible, however, that neither $k|f| \leqslant e$ nor $k|f| \geqslant e$ holds. In this case we have

$$p = (k|f| - e)^+ > 0 \text{ and } q = (k|f| - e)^- > 0.$$

Note that $q = e - \inf(k|f|, e)$. Now we have $A_q \neq L$. Indeed, if A_q were equal to L, then it would follow from $\inf(p, q) = 0$ that $p = 0$. On the other hand, $f \in I_L$ and $\inf(k|f|, e) \in A_f$ implies (by (1)) that

$$A_{e - \inf(k|f|, e)} = A_q = L.$$

We have obtained therefore a contradiction. This shows that $I_L \subset I$.

Conversely, we shall prove that $I \subset I_L$. For this purpose, let $f \in I$ and suppose that f is not a member of I_L. Again by (1), there exists then an element $h \in A_f$ such that $A_{e-h} \neq L$, so $A_{e-|h|} \neq L$ (since $|e - |h|\,| \leqslant |e-h|$). Now, $h \in A_f$ implies that $|h| \leqslant m|f|$ for some natural number m. Therefore,

$$0 \leqslant e - m|f| \leqslant e - |h|.$$

It follows that $A_{e-m|f|} \neq L$ and so $e \notin A_{e-m|f|}$. On the other hand $f \in I$ implies $2m|f| \leqslant e$, i.e.,

$$0 \leqslant e \leqslant 2(e - m|f|).$$

But this contradicts $e \notin A_{e-m|f|}$. Hence, I is a subset of I_L and the proof is complete.

It follows from this theorem that a Riesz space L with a strong unit is Archimedean if and only if the intersection of all maximal ideals in L consists of the zero element only. Hence, the Archimedean property in a Riesz space with a strong unit can be compared with the notion of semi-simplicity in a commutative ring with unit element. We recall that a commutative ring R with unit element is called semi-simple if the Jacobson radical of R consists of the zero element only. Since a Riesz space L is hyper-archimedean if and only if the quotient Riesz space L/A is Archimedean for every proper ideal A in L, it is a matter of course to ask whether a commutative ring R with unit element is regular if and only if the quotient ring R/A is semi-simple for every proper ideal A in R. This is indeed the case. For the proof we use the following theorem.

THEOREM 6. (L. LESIEUR, [8], *theorem* 10). *In a commutative ring R with unit element the following conditions are equivalent.*

(1) *R is regular.*

(2) *Every proper ideal in R is equal to the intersection of all maximal ideals containing the ideal.*

PROOF. (1) $\Rightarrow$ (2). Let A be a proper ideal. Then

$$\sqrt{A} = \cap\,(P : P \text{ proper prime ideal, } P \supset A).$$

(this property holds in an arbitrary commutative ring). From the regularity of R it follows that $\sqrt{A} = A$ and that

$$(P : P \text{ proper prime ideal, } P \supset A) = (J : J \text{ maximal ideal, } J \supset A).$$

Hence,

$$A = \cap\,(J : J \text{ maximal ideal, } J \supset A).$$

(2) $\Rightarrow$ (1). It is sufficient to prove that $(a) = (a^2)$ holds for all $a \in R$. This being evident if a is a unit, we may assume that (a) is a proper ideal. If J is a maximal ideal, then J is prime, so $a \in J$ if and only if $a^2 \in J$. It follows that

$$\begin{aligned}(a) &= \cap\,(J : J \text{ maximal ideal, } J \supset (a)) = \\ &= \cap\,(J : J \text{ maximal ideal, } J \supset (a^2)) = (a^2).\end{aligned}$$

COROLLARY. *A commutative ring R with unit element is regular if and only if R/A is semi-simple for every proper ideal A in R.*

PROOF. If R is regular and A is a proper ideal in R, then R/A is (as a homomorphic image of a regular ring) again a regular ring. Hence, R/A is semi-prime and the set of all proper prime ideals in R/A coincides with the set of all maximal ideals in R/A. It follows that the intersection of all maximal ideals in R/A consists of the zero element of R/A only, i.e., R/A is semi-simple.

Conversely, let R/A be semi-simple for all proper ideals A in R. It is sufficient to prove that

$$A = \cap\,(J : J \text{ maximal ideal, } J \supset A)$$

holds for every proper ideal A in R. Now, the semi-simplicity of R/A implies that

$$\cap\,(\bar{J} : \bar{J} \text{ maximal ideal in } R/A) = \{\bar{0}\},$$

where $\bar{0}$ is the zero of R/A. The desired result follows on account of the one-one correspondence between the maximal ideals in R containing A and the maximal ideals in R/A.

In the next theorem we shall prove the analogue of Lesieur's theorem for Riesz spaces with a strong unit.

THEOREM 7. *In a Riesz space L with a strong unit $e>0$ the following conditions are equivalent.*

(1) *L is hyper-archimedean.*

(2) *Every proper ideal in L is equal to the intersection of all maximal ideals containing the ideal.*

PROOF. (1) $\Rightarrow$ (2). In an arbitrary Riesz space L we have

$$A = \cap\,(P\colon P \text{ proper prime ideal, } P \supset A)$$

for every proper ideal A in L. But L is hyper-archimedean, so

$$(P\colon P \text{ proper prime ideal, } P \supset A) = (J\colon J \text{ maximal ideal, } J \supset A).$$

Hence,

$$A = \cap\,(J\colon J \text{ maximal ideal, } J \supset A).$$

(2) $\Rightarrow$ (1). It is sufficient to prove under the present hypothesis that L/A is Archimedean for all proper ideals A in L. It follows easily from

$$A = \cap\,(J\colon J \text{ maximal ideal, } J \supset A)$$

that

$$\cap\,(\bar{J}\colon \bar{J} \text{ maximal ideal in } L/A) = \{\bar{0}\},$$

where $\bar{0}$ denotes the zero of L/A. Since L/A has a strong unit $\bar{e}$, the only infinitely small element in L/A is, by theorem 5, the zero element, showing that L/A is Archimedean.

In view of theorems 6 and 7 it is a reasonable conjecture that a distributive lattice with smallest and largest element is a Boolean algebra if and only if every proper ideal in the lattice is equal to the intersection of all maximal ideals containing the ideal. Evidently, the latter condition is necessary for the lattice to be a Boolean algebra. This condition, however, is not sufficient, as shown by the following example (due to J. Varlet, who kindly permitted me to publish it here).

EXAMPLE. Let X be a distributive lattice with largest element e ($X \neq \{e\}$) but without a smallest element. Assume that X is relatively complemented, i.e., every closed interval $[a, b] = (c\colon c \in X, a \leqslant c \leqslant b)$ in X is a Boolean algebra. Let X_0 be the distributive lattice which is obtained by adjoining to X a smallest element θ. Note that X is a sublattice of X_0 but not an ideal in X_0.

An example of the situation as described in the above paragraph is the following. If $\mathscr{A}$ is an infinite point set, take for X_0 the collection consisting of the empty set, all finite subsets of $\mathscr{A}$ (for notational convenience denoted by $a, b, \ldots$) and $\mathscr{A}$ itself. Then X_0 is a distributive lattice with respect to partial ordering by anti-inclusion, i.e., $a \leqslant b$ if and only if $a \supset b$. Hence,

$$a \vee b = a \cap b \text{ and } a \wedge b = a \cup b.$$

The smallest element θ of X_0 is $\mathscr{A}$ and the largest element e of X_0 is the empty set. The collection X of all finite subsets of $\mathscr{A}$ (including the empty set) is, as a sublattice of X_0, a distributive lattice with largest element e but without a smallest element. Moreover, X is relatively complemented.

Observe now that $\{\theta\}$ is a proper prime ideal in X_0 but not a maximal ideal, so X_0 is not a Boolean algebra. We assert that every proper ideal in X_0 is equal to the intersection of all maximal ideals containing the ideal. First, we prove that every proper prime ideal $P \neq \{\theta\}$ in X_0 is a maximal ideal. Indeed, suppose that J is an ideal in X_0 that properly contains P. Then $a \notin P$ for some $a \in J$. For the proof that $J = X_0$ it is sufficient to show that $e \in J$. Take $r \in P$ such that $r \neq \theta$. Since X is relatively complemented and $[r, e]$ is a closed interval in X, it follows from $a \vee r \in [r, e]$ that there exists $p \in X$ such that

$$(a \vee r) \wedge p = r, \quad (a \vee r) \vee p = e.$$

Necessarily, p is a member of P. Indeed, since $a \notin P$ we have $a \vee r \notin P$, so $r = (a \vee r) \wedge p \in P$ implies $p \in P$. It follows from $a \in J$, $p \in J$ and $r \in J$ that $e = a \vee r \vee p \in J$.

As an immediate consequence we get that every proper ideal A in X_0, not consisting of θ only, is equal to the intersection of all maximal ideals containing A. Indeed, if $a \notin A$ and P is a prime ideal in X_0 containing A such that $a \notin P$, then $P \neq \{\theta\}$, so P is a maximal ideal. Therefore,

$$a \notin \cap\, (J : J \text{ maximal ideal in } X_0,\ J \supset A).$$

This implies that

$$A = \cap\, (J : J \text{ maximal ideal in } X_0,\ J \supset A).$$

It remains to prove that

$$I_{X_0} = \cap\, (J : J \text{ maximal ideal in } X_0) = \{\theta\}.$$

On account of the one-one correspondence between the maximal ideals in X_0 and the maximal ideals in X, it is sufficient for this purpose to prove that

$$I_X = \cap\, (J' : J' \text{ maximal ideal in } X)$$

is empty. Suppose on the contrary that I_X is non-empty and that $a \in I_X$. Since X does not have a smallest element, there exists $b \in X$, $b \notin [a, e]$, i.e., a is not a member of the principal ideal in X generated by b. Let P be a prime ideal in X maximal with respect to the property of not containing a. For any $x \notin P$, there exists an element $p \in P$ with the property that $p < x$. In order to prove this assertion, note that a belongs to the ideal in X generated by P and x and thus $a=p' \vee y$ for some $p' \in P$, $y \leqslant x$. This implies that $a \wedge x = p \vee y$ with $p = p' \wedge x \in P$, so

$$x = x \vee (a \wedge x) = x \vee (p \vee y) = x \vee p.$$

Since X is relatively complemented, there exists $q \in X$ satisfying

$$x \wedge q = p, \quad x \vee q = e.$$

But then $p \in P$, $x \notin P$ implies $q \in P$. Summarizing, for every $x \notin P$ there exists $q \in P$ such that $x \vee q = e$. It is an easy consequence that P is a maximal ideal in X. However, $a \notin P$ which is in contradiction to $a \in I_X$.

4. Necessary and sufficient conditions in order that every proper prime ideal contains a unique minimal prime ideal

In the present section we discuss another theorem that holds for commutative rings, for Riesz spaces and for distributive lattices with smallest element as well. The similarity in the proofs is again based on the fact that multiplication in a commutative ring behaves very similar to the infimum operation in a Riesz space or in a distributive lattice with smallest element. The theorem involved concerns necessary and sufficient conditions for every proper prime ideal to contain a unique minimal prime ideal.

In one of his papers (see [2], theorem 2.4) W. Cornish presents several conditions of this kind for a distributive lattice X with smallest element. Analogous conditions hold for a commutative semi-prime ring R with unit element and for a Riesz space L. In the first case we have to replace the infimum operation in X by multiplication in R and the notion of disjoint complement in X by the notion of annihilator in R. Because of the similarity in the proofs we present the proof only for R.

Let R be a commutative semi-prime ring. If P is a proper prime ideal in R, define $\mathcal{O}(P)$ by

$$\mathcal{O}(P) = (r : r \in R,\ rs = 0 \text{ for some } s \notin P).$$

Then $\mathcal{O}(P)$ is a subset of P and in fact an ideal in R. Note that if M is a minimal prime ideal in R, then $\mathcal{O}(M) = M$ (cf. [4], lemma 1.1). Moreover $\mathcal{O}(P) = \sqrt{\mathcal{O}(P)}$. Indeed, if $q \in \sqrt{\mathcal{O}(P)}$, then $q^k \in \mathcal{O}(P)$ for some natural number k, so $q^k s = 0$ for some $s \notin P$. Hence, $(qs)^k = 0$, i.e., $qs = 0$ since R is semi-prime. In other words, $q \in \mathcal{O}(P)$. For the proof of the main theorem of this section we first need two lemmas. In the proof of the first lemma we shall make use of the following theorem (cf. [7], lemma 3.1): if P is a proper prime ideal in a commutative ring containing an ideal A, then P is a minimal prime ideal with respect to A if and only if for every $r \in P$ there exists an element $s \notin P$ and a natural number k such that $r^k s \in A$.

Lemma A. *If P is a proper prime ideal in R and N is a minimal prime ideal with respect to $\mathcal{O}(P)$, then $N \subset P$.*

PROOF. If N were not a subset of P, there would exist an element $p \in N$, $p \notin P$. But then $p^k q \in \mathcal{O}(P)$ for some $q \notin N$ and some natural number k. Hence, $p^k qr = 0$ for some $r \notin P$. This, however, implies that $pqr = 0$, so it follows from $pr \notin P$ that $q \in \mathcal{O}(P)$ and therefore $q \in N$. We have obtained thus a contradiction. It follows that N is a subset of P.

LEMMA B. *If P is a proper prime ideal in R, then*

$$\mathcal{O}(P) = \cap\,(M : M \text{ minimal prime ideal, } M \subset P).$$

PROOF. If Q is a proper prime ideal such that $Q \subset P$, then $\mathcal{O}(P) \subset Q$. Indeed, $r \in \mathcal{O}(P)$ implies $rs = 0$ for some $s \notin P$. Now, if r were not a member of Q, then s would belong to Q, so $s \in P$, which is impossible. Hence $\mathcal{O}(P) \subset Q$. It follows that

$$(1) \qquad \mathcal{O}(P) \subset \cap\,(M : M \text{ minimal prime ideal, } M \subset P).$$

On the other hand,

$$\mathcal{O}(P) = \sqrt{\mathcal{O}(P)} = \cap\,(N : N \text{ minimal prime ideal with respect to } \mathcal{O}(P)).$$

If N is a minimal prime ideal with respect to $\mathcal{O}(P)$, then N is, by lemma A, a subset of P. Since every proper prime ideal in R contains a minimal prime ideal, we have $M_N \subset N$ for some minimal prime ideal M_N and thus $M_N \subset P$. By associating to every prime ideal N that is minimal with respect to $\mathcal{O}(P)$ such a minimal prime ideal M_N, we get

$$\begin{aligned}(2) \quad \mathcal{O}(P) &= \cap\,(N : N \text{ minimal prime ideal with respect to } \mathcal{O}(P)) \supset \\ &\supset \cap\,(M_N : N \text{ minimal prime ideal with respect to } \mathcal{O}(P)) \supset \\ &\supset \cap\,(M : M \text{ minimal prime ideal, } M \subset P).\end{aligned}$$

The inclusions (1) and (2) give the desired result.

THEOREM 8. (*cf.* W. H. CORNISH, [2], *theorem* 2.4 *for the analogous theorem in a distributive lattice with smallest element*). *Let R be a commutative semi-prime ring with unit element e. Then the following conditions are equivalent.*

(1) *If M_1, M_2 are different minimal prime ideals in R, then $R = M_1 + M_2$.*
(2) *Every proper prime ideal in R contains a unique minimal prime ideal.*
(3) *Every maximal ideal in R contains a unique minimal prime ideal.*
(4) *$\mathcal{O}(P)$ is prime for every proper prime ideal P in R.*
(5) *If $r, s \in R$ satisfy $rs = 0$, then $R = I(r) + I(s)$.*
(6) *If $r, s \in R$, then $I(rs) = I(r) + I(s)$.*

PROOF. The equivalence of (2) and (3) being evident, we shall prove (1) $\Rightarrow$ (2) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (5) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2). If P is a proper prime ideal and P contains two different minimal prime ideals M_1 and M_2, then $R = M_1 + M_2 \subset P$, which is impossible. Hence, P contains a unique minimal prime ideal.

(2) $\Rightarrow$ (4). If M is the unique minimal prime ideal contained in the proper prime ideal P, then, by lemma B, $\mathcal{O}(P) = M$. Therefore, $\mathcal{O}(P)$ is prime.

(4) $\Rightarrow$ (5). Let $r, s \in R$ satisfy $rs=0$ and assume that $I(r)+I(s)$ is a proper ideal. Then $I(r)+I(s)$ is contained in some maximal, and hence proper prime ideal J. Observe now that $I(r) \subset J$ implies $r \notin \mathcal{O}(J)$. Indeed, if r is a member of $\mathcal{O}(J)$, there exists $t \notin J$ satisfying $rt=0$. But then $t \in I(r) \subset J$, which is impossible. Similarly, we get $s \notin \mathcal{O}(J)$. Since, by hypothesis, $\mathcal{O}(J)$ is prime, we have on the one hand $rs \notin \mathcal{O}(J)$ and on the other hand $rs=0$. We have obtained therefore a contradiction. This shows that $R=I(r)+I(s)$.

(5) $\Rightarrow$ (6). Let $r, s \in R$ be arbitrary. Now, $I(r)+I(s) \subset I(rs)$ holds without any additional condition on R. It remains to prove, therefore, that $I(rs) \subset I(r)+I(s)$. For this purpose, let $q \in I(rs)$, i.e., $qrs=0$. Then it follows from (5) that $R=I(qr)+I(s)$, so

$$e=a+b, \ a \in I(qr) \text{ and } b \in I(s).$$

Hence,

$$q=qa+qb, \ qa \in I(r) \text{ and } qb \in I(s).$$

In other words, $q \in I(r)+I(s)$, which is the desired result.

(6) $\Rightarrow$ (5). If $r, s \in R$ and $rs=0$, then

$$R=I(0)=I(rs)=I(r)+I(s).$$

(5) $\Rightarrow$ (1). Let M_1 and M_2 be two different minimal prime ideals and let $r \in M_1$, $r \notin M_2$. From $r \in M_1$ it follows that $rs=0$ for some $s \notin M_1$ (cf. [4], lemma 1.1). Note now that $r \notin M_2$ implies $I(r) \subset M_2$. Similarly, $I(s) \subset M_1$. By hypothesis,

$$R=I(r)+I(s) \subset M_1+M_2,$$

so $R=M_1+M_2$. This completes the proof.

For the sake of completeness we conclude this section by stating (without proof) the corresponding theorem for Riesz spaces.

THEOREM 9. *Let L be a Riesz space. Then the following conditions are equivalent.*

(1) *If M_1, M_2 are different minimal prime ideals in L, then $L=M_1+M_2$.*
(2) *Every proper prime ideal in L contains a unique minimal prime ideal.*
(3) *$\mathcal{O}(P)=(f: f \in L, \inf(|f|, |g|)=0$ for some $g \notin P)$ is a prime ideal for every proper prime ideal P in L.*
(4) *If $u, v \in L^+$ and $\inf(u, v)=0$, then $L=\{u\}^d+\{v\}^d$.*
(5) *If $u, v \in L^+$, then $\{\inf(u, v)\}^d=\{u\}^d+\{v\}^d$.*
Moreover, if L has a strong unit, then each of these conditions is equivalent to
(6) *Every maximal ideal in L contains a unique minimal prime ideal.*

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